Melting Block 4

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The original Melting Block

The original Melting Block was designed by Tom O'Beirne. It is a paradoxical 3-dimensional packing puzzle. It uses pieces which are based on a rectangular solid 'brick' of dimensions 19x29x44. When 27 of these bricks are stacked in a 3x3x3 arrangement, with all of the bricks oriented the same way, the result is a rectangular solid of size 57x87x132. However, if you take 28 of these same bricks and stack them in a 2x2x7 arrangement, with the smallest dimension being stacked 7 deep, then the resulting rectangular solid has dimensions 58x88x133.

27 bricks, when packed in the slightly larger 58x88x133 box will look to fill the box with virtually no room to spare, but by rearranging all the pieces, a 28th brick can be added.

Tom disguises the basic design of the puzzle by using only 8 pieces, with an additional copy of the smallest one. The pieces are size: 19x29x44 (2 copies), 38x29x44, 19x58x44, 19x29x88, 38x58x44, 38x29x88, 19x58x88 and 38x58x88. The dimensions of each of these pieces are either 1 or 2 times the basic 'brick' dimensions in each direction. The 3x3x3 packing is very simple, with the assembled puzzle split into the 8 pieces by cuts made 1/3 of the way along in each direction. The 2x2x7 packing is a little more interesting and challenging to find.

Melting Block 4

John Rausch got me interested in finding a similar, but different design. I had many years ago noticed that 4x4x4 = (3x3x7)+1, so that this pairing could give rise to a similar paradoxical design, but had not come up with an obvious choice for the pieces.

The calculation of the size of the basic building block used for this larger puzzle is relatively easy:

Let x, y and z be the dimensions of this 'brick', where x < y < z.

When packed in a 4x4x4 arrangement, the dimensions of the resulting rectangular solid will be 4x by 4y by 4z. When packed in the 3x3x7 fashion, with the smallest dimension x being stacked 7 deep, the resulting rectangular solid has dimensions 3y by 3z by 7x. We want the 4x4x4 solid to be one unit larger in every direction than the 3x3x7 formation.

Hence, we get the equations:

$$4x = 3y + 1$$
$$4y = 3z + 1$$
$$4z = 7x + 1$$

Or:

$$4x - 3y \quad = 1$$
$$4y - 3z = 1$$
$$-7x \quad + 4z = 1$$

The solution to this system of linear equations is x=37, y=49, and z=65.

The box sizes are: small = 147 x 195 x 259 and large = 148 x 196 x 260.

The First Analysis and Design

I decided that all the pieces should be rectangular solids and different (except for the possibly duplicated single brick to be added for the larger solution). If you allow each piece to be 1, 2, or 3 times the basic brick dimension in each direction, that gives us 27 different pieces. The total volume of these 27 pieces is 216 'bricks', or much more than the 64 'bricks' needed. Hence, the problem reduces to finding a subset of these 27 pieces which can be assembled to form both solids. I was hoping some of the solutions would have unique solutions to one or both of the packing problems.

A computer analysis was run which counted all the solutions, and also kept track of which subsets of the 27 pieces were used.

It took about two hours to analyze the solutions to the 3x3x7 box:

> - 26,302,424 solutions (after R&R reduction)
> - 51,024 different subsets of the pieces were used
> - 770 of these subsets had a unique solution

It took another two hours to analyze the 4x4x4 box. A second copy of the single brick was allowed:

> - 33,032,152 solutions
> - 85,397 different subsets used
> - 4,199 of these subsets had a unique solution

I then ran a program to compare the results. I was elated to find 4 subsets that had unique solutions to both size boxes. In all the cases, the smaller box used a single brick and the larger box used a second single brick. These solutions are called 'doubly-unique' – there is a unique solution to the smaller box (with a single small brick) and a unique solution to the larger box (using two copies of the single brick).

These packing solutions are much harder to find than the solutions to the original Melting Block.

Two of the subsets use 11 pieces for the smaller box and 12 for the larger one. The other two subsets use 12 pieces for the smaller box and 13 for the larger one. On further inspection, the two 11 piece subsets are essentially the same – the two larger dimensions are switched between the two. The same similarity goes for the two 12 piece solutions. This 'duality' is discussed in more detail later.

For my design, I chose one of the 11 piece subsets – the packing solutions look nicer to me. I found the 12 piece solutions harder to find, but 'niceness' won out on this design.

Both doubly unique 11-piece solutions:

```
 37 x   49 x   65  X 2   37 x   49 x   65  X 2
 74 x   49 x   65         74 x   49 x   65
111 x   49 x   65        111 x   49 x   65
 74 x   98 x   65         74 x   49 x 130
 37 x  147 x   65         37 x   49 x 195
111 x  147 x   65        111 x   49 x 195
111 x   49 x 130        111 x   98 x   65
 74 x   98 x 130         74 x   98 x 130
 37 x  147 x 130         37 x   98 x 195
 37 x   49 x 195         37 x  147 x   65
111 x   98 x 195        111 x  147 x 130
```


Both doubly unique 12-piece solutions:

```
 37 x   49 x   65 X 2    37 x   49 x   65  X 2
111 x   49 x   65        111 x   49 x   65
 37 x   98 x   65         37 x   49 x 130
 74 x   98 x   65         74 x   49 x 130
 37 x  147 x   65         37 x   49 x 195
 74 x  147 x   65         74 x   49 x 195
 74 x   49 x 130         74 x   98 x   65
111 x   49 x 130        111 x   98 x   65
 37 x   98 x 130         37 x   98 x 130
111 x  147 x 130        111 x   98 x 195
 37 x   49 x 195         37 x  147 x   65
111 x   49 x 195        111 x  147 x   65
```


The results of this analysis, including a model of the 11-piece solution, were presented at the New York Puzzle Party on February 13, 2016.

The Full Analysis

John suggested that we also allow rectangular solids in which the bricks are stacked up to 4 deep in the smallest dimension. Such pieces would also fit into the 3x3x7 box. Stacking bricks 4 deep in the other two dimensions would create pieces that will not fit into the 3x3x7 box.

This increases the number of pieces from 27 to 36. The same analysis was run on this larger set of pieces.

It took 2-1/2 hours to analyze the solutions to the 3x3x7 box:

- 110,410,807 solutions (after R&R reduction)
- 435,158 different subsets of the pieces were used
- 14,999 of these subsets had a unique solution

It took 12 hours to analyze the 4x4x4 box, with a second single brick allowed:

- 385,302,069 solutions
- 987,060 different subsets used
- 58,983 of these subsets had a unique solution

The program to compare the results produced:

- 204 doubly-unique solutions
- 106 of the doubly-unique solutions use two copies of the single brick
- 98 of the doubly-unique solutions use only one single brick (in the 4x4x4 box)

We decided we preferred to use the solutions which include only one copy of the single brick, used in the 4x4x4 box.

We also decided not to use any of the 3 rectangular solids which are 4 bricks thick in the smallest dimension and 3 bricks thick in the middle dimension. These solids have unit size 148 x 147 x n, and it is difficult to determine the difference between the first two dimensions. The two larger such pieces do not appear in any of the doubly-unique solutions, but the smallest such piece is in 22 of the 98 single-brick doubly-unique solutions.

The Multipuzzle

We propose an MB4 'multipuzzle', in which there are 28 pieces labelled with a number or letter, and a list of 76 problems. Each problem specifies a list of from 10 to 13 of these pieces, plus the single brick. Each of these have doubly-unique solutions.

The appendix at the end of this document has tables of the 28 pieces and 76 problems.

The CRUMB Puzzle

'CRUMB' stands for "Cutler Rausch Unique Melting Block". One of the 11/12 piece doubly-unique solutions was selected as an entry in the 2016 Puzzle Design Competition. There is additional information in the Appendix.

Duality

MB4 has an interesting 'duality' involving the interchange of the two larger dimensions of the brick:

Piece Duality

Each of the 36 pieces in the second analysis has a dual. For example, take piece #7, which is constructed from 6 bricks in a 3x2x1 stack – 3 units deep in the thinnest dimension; two units deep in the middle dimension, and one unit deep in the thickest dimension, for a total size of 111 x 98 x 65. To create its dual, switch the number of bricks stacked in the two thicker dimensions. The stack is now 3x1x2, for a total size of 111 x 49 x 130. This is piece #F.

Hence, pieces 7 and F are dual. If the first two dimensions of the stack are the same, then the piece is 'self-dual', for example pieces 4 (4x1x1) and H (1x2x2).

Note this duality only works when switching the last two dimensions – the first (smallest) dimension allows for stacking bricks 4 deep.

Solution Duality

Any solution to the 4x4x4 box or the 3x3x7 box has a 'dual' solution. To construct this solution, do the following:

For each piece in the original solution, take the 'dual' of this piece, and use it in the new solution with the same location, except with the y and z locations of the piece switched, where y is the direction where the middle brick dimension extends, and z is the direction where the largest brick dimension extends. For example, if piece #5 fills locations (3,2,1) and (3,3,1) in the original solution, then the dual piece D will fill locations (3,1,2) and (3,1,3) in the dual solution.

[example with pictures needed here]

Piece-Set Duality

Each solution uses a particular set of the 36 pieces. If the piece-set which creates a particular solution is called 'a', then the dual solution uses the piece set which is the dual of 'a'. To construct the dual piece-set, make the following changes:

- Pieces in the piece-set that are self-dual remain the same
- If a piece in the piece-set is not self-dual, and its dual piece is not in the piece-set, then replace this piece with its dual
- If a piece in the piece-set is not self-dual, but its dual is also in the piece-set, then they both remain

If no changes are made to the piece-set, then the piece-set is self-dual.

If a piece-set has m solutions to the 3x3x7 box and n solutions to the 4x4x4 box (with an additional single brick added), then the dual piece-set will also have m and n solutions, respectively. If a piece-set has doubly-unique solutions to the two boxes, then its dual piece-set will also have doubly-unique solutions.

Duality in the 76-Solution Multipuzzle

We removed piece #C from the set of pieces used for the multipuzzle, because it is 'almost square'. Its brick dimensions are 4x3x1 and the exact dimensions are 148x147x65. However, its dual, piece #S, is not 'almost square'. Its brick dimensions are 4x1x3 and the exact dimensions are 148x49x195. 22 of the 76 doubly-unique solutions use piece #S. The dual piece-sets would contain piece #C, and are not included. The remaining 54 doubly-unique solutions consist of 27 pairs of dual piece sets.

Table of Pieces used in the Full Analysis

piece name	brick dims	brick volume	physical dimensions			use count	piece dual	used in multipuzzle
1	1 1 1	1	37	49	65	0	1	yes
2	2 1 1	2	74	49	65	49	2	yes
3	3 1 1	3	111	49	65	49	3	yes
4	4 1 1	4	148	49	65	51	4	yes
5	1 2 1	2	37	98	65	45	D	yes
6	2 2 1	4	74	98	65	40	E	yes
7	3 2 1	6	111	98	65	33	F	yes
8	4 2 1	8	148	98	65	30	G	yes
9	1 3 1	3	37	147	65	32	P	yes
A	2 3 1	6	74	147	65	35	Q	yes
B	3 3 1	9	111	147	65	31	R	yes
C	4 3 1	12	148	147	65	0	S	square
D	1 1 2	2	37	49	130	44	5	yes
E	2 1 2	4	74	49	130	46	6	yes
F	3 1 2	6	111	49	130	28	7	yes
G	4 1 2	8	148	49	130	25	8	yes
H	1 2 2	4	37	98	130	50	H	yes
I	2 2 2	8	74	98	130	24	I	yes
J	3 2 2	12	111	98	130	20	J	yes
K	4 2 2	16	148	98	130	0	K	unused
L	1 3 2	6	37	147	130	36	T	yes
M	2 3 2	12	74	147	130	14	U	yes
N	3 3 2	18	111	147	130	3	V	yes
O	4 3 2	24	148	147	130	0	W	sq&unused
P	1 1 3	3	37	49	195	31	9	yes
Q	2 1 3	6	74	49	195	29	A	yes
R	3 1 3	9	111	49	195	33	B	yes
S	4 1 3	12	148	49	195	22	C	yes
T	1 2 3	6	37	98	195	41	L	yes
U	2 2 3	12	74	98	195	16	M	yes
V	3 2 3	18	111	98	195	5	N	yes
W	4 2 3	24	148	98	195	0	O	unused
X	1 3 3	9	37	147	195	0	X	unused
Y	2 3 3	18	74	147	195	0	Y	unused
Z	3 3 3	27	111	147	195	0	Z	unused
a	4 3 3	36	148	147	195	0	a	sq&unused

Box Sizes: small = 147 x 195 x 259 and large = 148 x 196 x 260.
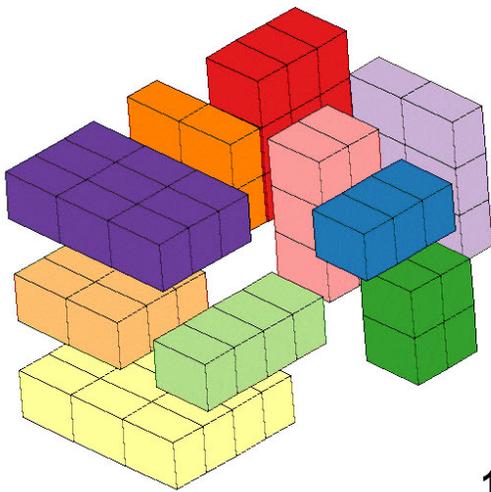
Multipuzzle – 76 Problems

```
        number      pieces used      dual of problem   status of dual
          --     ----------------   ----------------   ------------------
           1     10-2589DEJMST      10-256CDGJLPU      dual has piece C
           2     10-278DEFIPSU      10-25679CFGIM      dual has piece C
CRUMB 3          10-346ABFHLRS      10-347BCEHQRT      dual has piece C
video 4          10-679DEIJQST      10-56ACEFIJLP      dual has piece C
           5     11-234567DHMSU     11-2345CDEFHMU     dual has piece C
           6     11-23456AGHLQV     11-2348ADEHNQT     dual is number   10
           7     11-2345789ILRU     11-234BDFGIMPT     dual is number   11
           8     11-23458EHJLST     11-2346CDGHJLT     dual has piece C
           9     11-2346789JLRT     11-234BEFGJLPT     dual is number   12
          10     11-2348ADEHNQT     11-23456AGHLQV     dual is number    6
          11     11-234BDFGIMPT     11-2345789ILRU     dual is number    7
          12     11-234BEFGJLPT     11-2346789JLRT     dual is number    9
          13     11-23579EHLPSV     11-2369CDFHNPT     dual has piece C
          14     11-2359AEHLRSU     11-236BCDHMPQT     dual has piece C
          15     11-23678ABHLRT     11-23BEFGHLQRT     dual is number   16
          16     11-23BEFGHLQRT     11-23678ABHLRT     dual is number   15
          17     11-245679BJLRT     11-24BDEFJLPRT     dual is number   27
          18     11-245689AFHQV     11-247ADEGHNPQ     dual is number   25
          19     11-2456BFHIMQT     11-247ADEHILRU     dual is number   26
          20     11-24589BDHIRU     11-245BDGHIMPR     dual is number   22
          21     11-2458ADHILRU     11-245BDGHIMQT     dual is number   23
          22     11-245BDGHIMPR     11-24589BDHIRU     dual is number   20
          23     11-245BDGHIMQT     11-2458ADHILRU     dual is number   21
          24     11-24679DEIJST     11-2456CEFIJLP     dual has piece C
          25     11-247ADEGHNPQ     11-245689AFHQV     dual is number   18
          26     11-247ADEHILRU     11-2456BFHIMQT     dual is number   19
          27     11-24BDEFJLPRT     11-245679BJLRT     dual is number   17
          28     11-2568AEFHLRU     11-267BDEGHMQT     dual is number   35
          29     11-2568BEFHMQT     11-267ADEGHLRU     dual is number   34
          30     11-256BDGHIMPR     11-2589BDEHIRU     dual is number   33
          31     11-2578BEGHILQ     11-268ADFGHIRT     dual is number   36
          32     11-2578DFHJPST     11-2579CDFGHJL     dual has piece C
          33     11-2589BDEHIRU     11-256BDGHIMPR     dual is number   30
          34     11-267ADEGHLRU     11-2568BEFHMQT     dual is number   29
          35     11-267BDEGHMQT     11-2568AEFHLRU     dual is number   28
          36     11-268ADFGHIRT     11-2578BEGHILQ     dual is number   31
          37     11-34569AEHNRT     11-346BDEHLPQV     dual is number   47
          38     11-3456ADEIJST     11-3456CDEIJLQ     dual has piece C
          39     11-34579EIJPQS     11-3469ACDFIJP     dual has piece C
          40     11-34579EIJQRT     11-346ABDFIJLP     dual is number   45
```

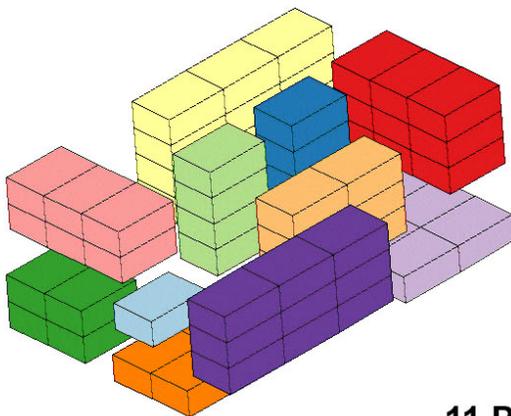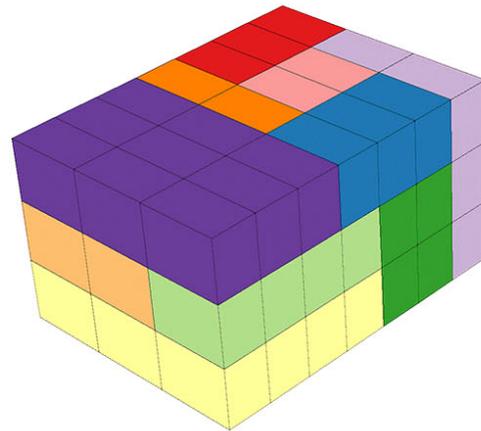| number	pieces used	dual of problem	status of dual
41	11-34589EGHJQR	11-3468ABDGHJP	dual is number 44
42	11-34589EHIJQR	11-346ABDGHIJP	dual is number 46
43	11-3467ADEHSTU	11-3456CEFHLMQ	dual has piece C
44	11-3468ABDGHJP	11-34589EGHJQR	dual is number 41
45	11-346ABDFIJLP	11-34579EIJQRT	dual is number 40
46	11-346ABDGHIJP	11-34589EHIJQR	dual is number 42
47	11-346BDEHLPQV	11-34569AEHNRT	dual is number 37
48	11-3569ADEPSTV	11-3569CDELNPQ	dual has piece C
49	11-356BEFGLMPQ	11-36789ADERTU	dual is number 50
50	11-36789ADERTU	11-356BEFGLMPQ	dual is number 49
51	12-2345679AGHMR	12-2348BDEFHPQU	dual is number 59
52	12-234567EGLPRU	12-234689BDEFMT	dual is number 57
53	12-2345689EFRTU	12-23467BDEGLMP	dual is number 56
54	12-23457AEGHJLT	12-23468DFHJLQT	dual is number 58
55	12-234589BDFLST	12-23457CDGLPRT	dual has piece C
56	12-23467BDEGLMP	12-2345689EFRTU	dual is number 53
57	12-234689BDEFMT	12-234567EGLPRU	dual is number 52
58	12-23468DFHJLQT	12-23457AEGHJLT	dual is number 54
59	12-2348BDEFHPQU	12-2345679AGHMR	dual is number 51
60	12-235678ABEHLR	12-236BDEFGHQRT	dual is number 62
61	12-2359BEHLPRST	12-2369BCDHLPRT	dual has piece C
62	12-236BDEFGHQRT	12-235678ABEHLR	dual is number 60
63	12-2456ADFHJPQS	12-24579ACDEHJQ	dual has piece C
64	12-345789AEHIPS	12-3469CDFGHIPQ	dual has piece C
65	12-345ADEHIPRST	12-34569BCDHILQ	dual has piece C
66	12-3467ADEFHLQS	12-34567ACEFHQT	dual has piece C
67	12-3567AGHLPQRT	12-389ABDEFHLQT	dual is number 69
68	12-3579DEGILPST	12-35689CDFILPT	dual has piece C
69	12-389ABDEFHLQT	12-3567AGHLPQRT	dual is number 67
70	12-45679ADFPRST	12-4579BCDEFLPQ	dual has piece C
71	12-4569BDFLMPQT	12-4579ADELPRTU	dual is number 72
72	12-4579ADELPRTU	12-4569BDFLMPQT	dual is number 71
73	13-2345679AGHQRT	13-2348ABDEFHLPQ	dual is number 74
74	13-2348ABDEFHLPQ	13-2345679AGHQRT	dual is number 73
75	13-345789AFHLPQT	13-3479ADFGHLPQT	dual is number 76
76	13-3479ADFGHLPQT	13-345789AFHLPQT	dual is number 75

CRUMB Solution

The smallest piece (made from darker wood) is the basic 'building block' of this puzzle. All of the other pieces are constructed by 'gluing' together multiple copies of this block, all oriented the same. The packing which does not use this piece uses 63 of these blocks in a 3x3x7 grid. The packing which uses this piece, has 64 blocks in a 4x4x4 grid. The solution to each packing is unique (not counting R&R of the entire box).

To solve either packing, use the small block as a guide. First, determine how this block should be oriented in the box. Then line up the other pieces so their 'building blocks' are all oriented the same way. When done correctly, each dimension of each piece will be 1, 2, 3 or 4 times the corresponding dimension of the small block. After this has been done, *do not rotate any of the pieces*. The desired solution will be found by translating these pieces to the box in the correct fashion. Knowing the solution is unique can be helpful in ruling out certain situations, such as two of the pieces forming a larger rectangular solid.

10 Pieces

11 Pieces